

Sample Exam 4

Short Answer

1. A metal bar of length 1 m travels through a perpendicular B field with magnitude 1 T. How fast would the bar need to go to develop an emf of 1 V across its ends.

$$\phi_B = l B x$$

$$\varepsilon = -\frac{d\phi_B}{dt} = -l B \frac{dx}{dt}$$

$$\varepsilon = -l B v$$

$$|v| = \frac{\varepsilon}{l B} = \frac{1V}{1m \cdot 1T} = 1m/s$$

2. A uniform magnetic field passes through a square loop of wire with 5 turns. The loop has side 0.1 m and is oriented at 30 degrees with respect to the field. What is the flux through the wire? If the field drops from 5 T to 4 T in 1 minute, what emf will develop and in what direction will it point.

$$\phi_{1loop} = BA \cos \theta = 5T \cdot (0.1)^2 \cdot \cos 30 = 0.0433 T m^2$$

$$\phi = 5 \cdot \phi_{1loop} = 0.217 T m^2$$

$$|\varepsilon| = \frac{d\phi}{dt} = 5 \cdot \frac{(5T - 4T) \cdot (0.1)^2 \cdot \cos 30}{60s} = 7.22 \times 10^{-4} V$$

3. A cylindrical solenoid is 0.5 m long and has 50 turns per cm. What field is present if 1 amp flows through the wire? (Hint: Calculate using Ampere's Law). What is the energy density?

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$$

$$Bl = \mu_0 i n l$$

$$B = \mu_0 i n = 4\pi \times 10^{-7} \cdot 1A \cdot \frac{50}{0.01m} = 6.283 \times 10^{-3} T$$

$$u = \frac{1}{2\mu_0} B^2 = \frac{1}{2 \cdot 4\pi \times 10^{-7}} \cdot (6.283 \times 10^{-3} T)^2$$

4. Compute the inductance of the solenoid in SA 3.

$$L = \frac{\phi}{i} = \frac{NBA}{i} = \frac{n l \mu_0 i n A}{i} = \mu_0 n^2 l A$$

$$L = 4\pi \times 10^{-7} \cdot \left(\frac{50}{0.01m}\right)^2 \cdot 0.5m \cdot A$$

5. A 100mH inductor has a current given by $i=2t^2+3t+4$. What voltage develops across the inductor at $t=3$ sec.

$$V_L = L \frac{di}{dt} = L \cdot (4t + 3) = 100 \times 10^{-3} \cdot (4 \cdot 3 + 3) = 1.5V$$

6. Write Maxwell's equations and briefly explain each equation.

$$(a) \oint \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0}$$

$$(b) \oint \vec{E} \cdot d\vec{r} = -\frac{d\phi_B}{dt}$$

$$(c) \oint \vec{B} \cdot d\vec{A} = 0$$

$$(d) \oint \vec{B} \cdot d\vec{r} = \mu_0 i_{encl} + \mu_0 \epsilon_0 \frac{d\phi_E}{dt}$$

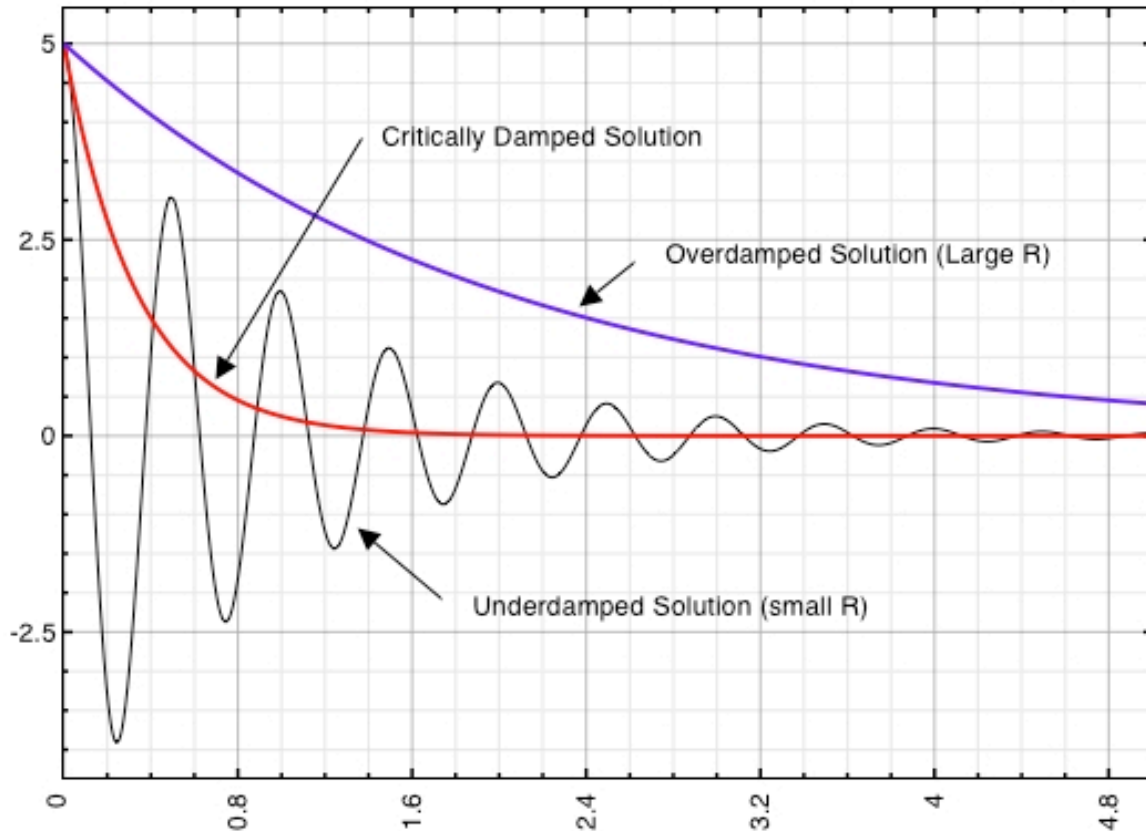
(a) Gauss' Law for E: The total flux of E through an enclosed surface is proportional to the charge enclosed

(b) A changing magnetic flux induces an electric field.

(c) Gauss' Law for B: There is no free magnetic charge.

(d) A magnetic field is produced by either a current or a changing electric flux.

7. Describe the underdamped, critically damped, and overdamped solutions of the RLC circuit.



8. An emf with frequency 60 Hz and $V_0 = 300$ drives a circuit with resistance 100 Ohms and inductance 0.5 H. What are the RMS voltage, the inductive reactance X_L , the impedance Z , and the maximum and RMS current.

$$V_{RMS} = \frac{V_0}{\sqrt{2}} = 212.1V$$

$$X_L = \omega L = 2\pi f L = 2\pi \cdot 60Hz \cdot 0.5H = 60\pi \Omega = 188.5 \Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100\Omega)^2 + (188.5\Omega - 0)^2} = 213.4 \Omega$$

$$i_{RMS} = \frac{V_{RMS}}{Z} = \frac{212.1V}{213.4 \Omega} = 0.994 A$$

$$i_0 = \sqrt{2} i_{RMS} = 1.41A$$

10. An incoming radio wave has a frequency f of 780 KHz. If the inductor in an LC circuit is 100mH, what should the capacitance be set to to tune this radio to this frequency.

$$f = 780 \times 10^3 \text{ Hz}$$

$$2\pi f = \sqrt{\frac{1}{LC}}$$

$$C = \frac{1}{L(4\pi^2 f^2)} = \frac{1}{100 \times 10^{-3} (4\pi^2 \cdot (780 \times 10^3 \text{ Hz})^2)} = 4.16 \times 10^{-13} \text{ F}$$

10. Derive the expression for the magnitude of the magnetic field at a distance r from a long straight wire.

We use Ampere's Law

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$$B \cdot (2\pi r) = \mu_0 i$$

$$B = \frac{\mu_0 i}{2\pi r}$$

11. A current of 10A is distributed uniformly across a beam with a radius a . Find the magnetic field at a distance r from the center of the beam. Consider both $r < a$ and $r > a$.

We can now use Ampere's Law (J is the current density: current /area)

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

For $r < a$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$$B \cdot (2\pi r) = \mu_0 J \pi r^2$$

$$B = \frac{\mu_0 J r}{2}$$

For $r \geq a$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 i$$

$$B \cdot (2\pi r) = \mu_0 J \pi a^2$$

$$B = \frac{\mu_0 J a^2}{2r}$$

Problems.

1. A magnetic field is given by

$$B = B_0 \cos \omega t$$

Note that the magnitude of B changes, but not its direction. A circular loop is placed in this field and oriented at an angle θ

a. What is the magnetic flux through the loop?

$$\varphi = BA \cos \theta$$

b. What emf is induced? Draw a picture to indicate the direction of the induced current

$$\begin{aligned}\varepsilon &= -\frac{d\varphi_B}{dt} \\ &= -\pi r^2 \frac{dB}{dt} \cos \theta \\ &= -\pi r^2 (-\omega B_0 \sin \omega t) \cos \theta \\ &= \pi r^2 \omega B_0 \sin \omega t \cos \theta\end{aligned}$$

The emf oscillates back and forth! The direction changes back and forth, always flowing to produce a B to oppose the change.

c. What angle leads to the maximum induced emf?

$$\theta = 0 \text{ leads to maximum change in flux}$$

d. Assume that θ is now 0. What electric field is present at the radius of the wire?

$$\begin{aligned}\oint \vec{E} \cdot d\vec{l} &= -\frac{d\varphi_B}{dt} \\ E(2\pi r) &= -\pi r^2 \frac{dB}{dt} \\ E &= -\frac{r}{2} \cdot (-\omega B_0 \sin \omega t) \\ E &= \frac{r}{2} \cdot (\omega B_0 \sin \omega t)\end{aligned}$$

2. A circular parallel plate capacitor with radius R and separation d in an RC circuit discharges via the equation

$$q = q_0 e^{-t/RC}$$

a. What is the electric field as a function of time?

$$V = \frac{q}{C} = \frac{q_0 e^{-t/RC}}{C} \quad C = \frac{\epsilon_0 A}{d}$$

$$E = \frac{V}{d} = \frac{q_0 e^{-t/RC}}{Cd}$$

b. What is the displacement current at $a/2$ and a ?

$$i_d = \epsilon_0 \frac{d\phi_E}{dt}$$

$$= \epsilon_0 (\pi a^2) \frac{dE}{dt}$$

$$= \epsilon_0 (\pi a^2) \cdot \frac{q_0 e^{-t/RC}}{Cd} \cdot \frac{-1}{RC} \quad \text{at radius } a$$

$$= \epsilon_0 \left(\pi \frac{a^2}{4}\right) \cdot \frac{q_0 e^{-t/RC}}{Cd} \cdot \frac{-1}{RC} \quad \text{at radius } a/2$$

c. What is the induced B at $a/2$ and a ?

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i_d$$

$$B(2\pi a) = \mu_0 i_d$$

$$B = \frac{\mu_0 i_d}{(2\pi a)} \quad \text{at radius } a$$

$$B = \frac{\mu_0 i_d}{(2\pi \frac{a}{2})} \quad \text{at radius } r/2$$

Note: You need to plug in the correct displacement current from part b.

3. Consider an RLC circuit with $R=10$ Ohms, $L = 500$ mH and $C=1$ microF.

a. Write the differential equation that describes this circuit by using Kirchoff's voltage loop rule. What is the natural osc. frequency?

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{500 \times 10^{-3} H \cdot 1 \times 10^{-6} F}} = 1414.2$$

b. Now consider driving this circuit. What are X_L , X_C , and Z for this circuit? Sketch the amplitude of the voltage across the capacitor as a function of the frequency.

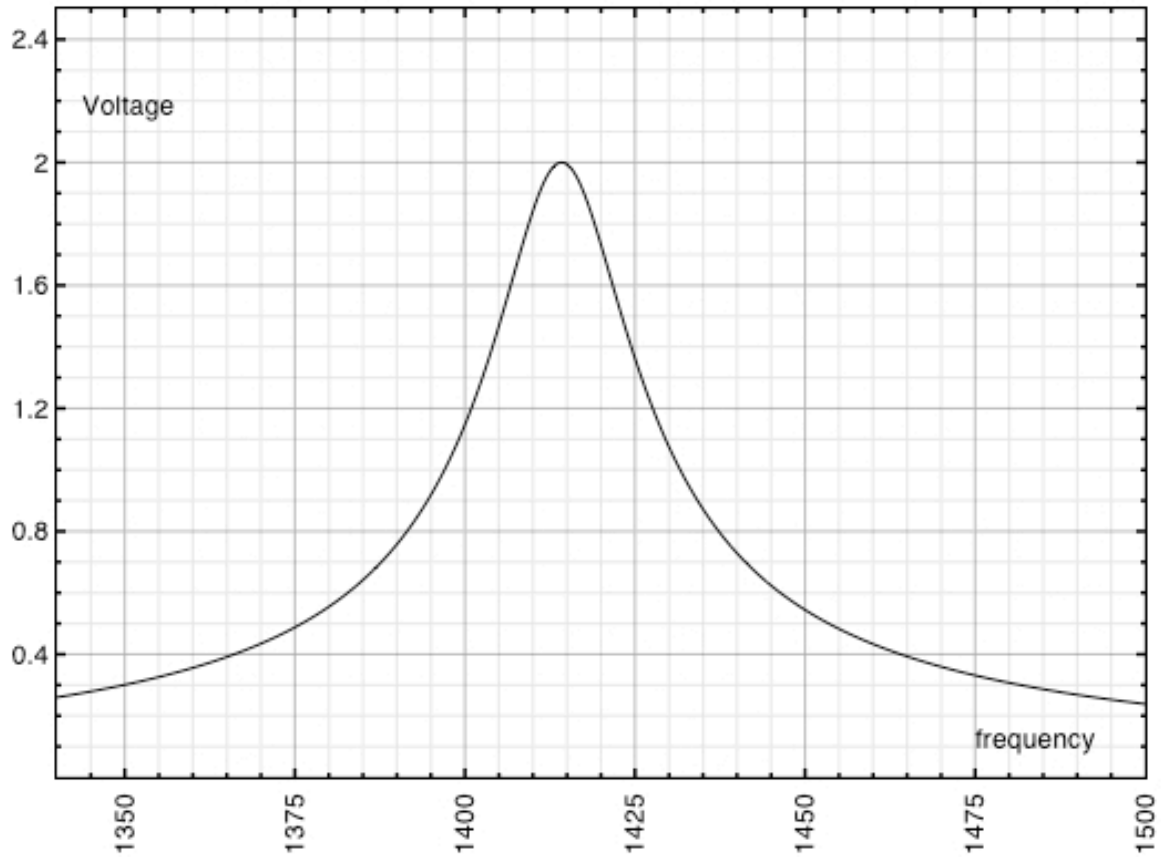
$$X_L = \omega L = 2\pi f L$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

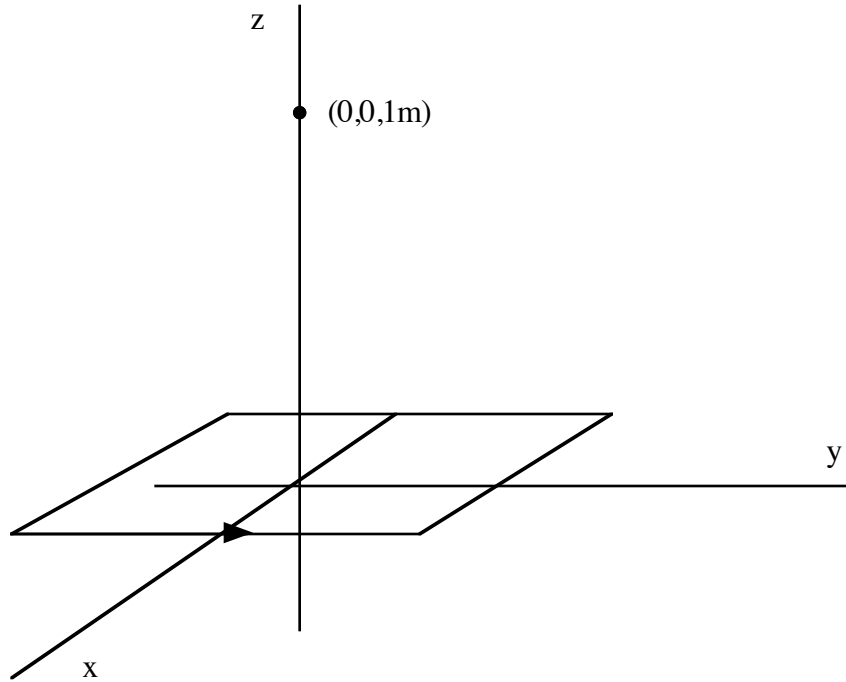
$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

Take the emf to be 20 for the purpose of plotting a nice curve. Here y is the voltage and x is the frequency.

$$y = \frac{20.0}{\sqrt{\left(10^2 + \left(x \cdot 0.5 - \frac{1}{x \cdot 1E-6}\right)^2\right)}}$$



3. A Square loop with side 1 m contains a single turn of wire. It is placed so that the center of the square is at the origin. Compute the magnetic field at a point $(0,0,1\text{m})$. Take the current to be 5 Amps and flowing in the counterclockwise direction as viewed from above.



The field can be computed by finding the z component due to just one of the four segments and then multiplying by 4. By symmetry, only the z component survives. We begin by computing the field using Biot-Savart for the *right* wire shown above. First, write out all the pieces we need.

$$d\vec{l} = dx \hat{i}$$

$$\vec{r} = (0-x)\hat{i} + (0-\frac{l}{2})\hat{j} + (z-0)\hat{k}$$

$$r = \sqrt{(0-x)^2 + (0-\frac{l}{2})^2 + (z-0)^2}$$

$$\hat{r} = \frac{-x\hat{i} - \frac{l}{2}\hat{j} + z\hat{k}}{\sqrt{x^2 + (\frac{l}{2})^2 + z^2}}$$

Note that the right wire is at $y=l/2$. The direction of the current will be given by the order of integration (from $x=+l/2$ to $x=-l/2$).

Now substitute in

$$\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{i d\vec{l} \times \hat{r}}{r^2} \\
&= \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{i(dx\hat{i}) \times \left(\frac{-x\hat{i} - \frac{l}{2}\hat{j} + z\hat{k}}{\sqrt{x^2 + (\frac{l}{2})^2 + z^2}} \right)}{x^2 + (\frac{l}{2})^2 + z^2} \\
&= \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{i(dx\hat{i}) \times (-x\hat{i} - \frac{l}{2}\hat{j} + z\hat{k})}{(x^2 + (\frac{l}{2})^2 + z^2)^{3/2}}
\end{aligned}$$

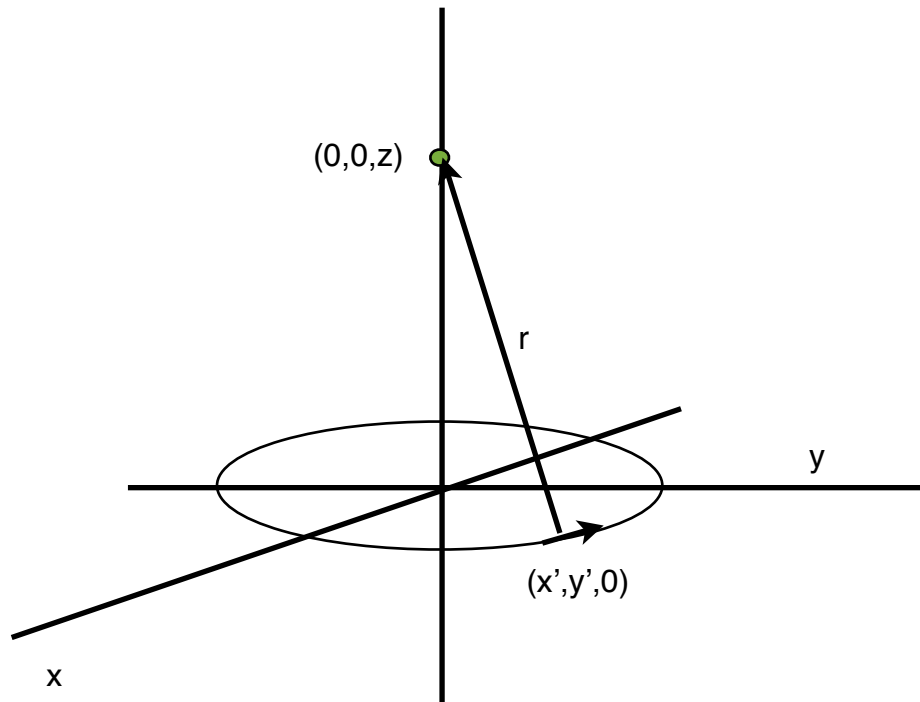
We now pick out just the z component (from $\hat{i} \times \hat{j}$)

$$\begin{aligned}
B &= \frac{\mu_0}{4\pi} \int_{-l/2}^{l/2} \frac{i(dx\hat{i}) \times (-x\hat{i} - \frac{l}{2}\hat{j} + z\hat{k})}{(x^2 + (\frac{l}{2})^2 + z^2)^{3/2}} \\
B_z &= \frac{\mu_0 i}{4\pi} \int_{-l/2}^{l/2} \frac{(-\frac{l}{2} dx \hat{k})}{(x^2 + (\frac{l}{2})^2 + z^2)^{3/2}} \\
&= -\frac{\mu_0 i l}{8\pi} \int_{-l/2}^{l/2} \frac{dx}{(x^2 + (\frac{l}{2})^2 + z^2)^{3/2}} \\
&= \frac{\mu_0 i l}{8\pi} \cdot \frac{8 l \sqrt{(\frac{l}{2})^2 + z^2}}{(l^2 + 2z^2) \cdot (l^2 + 4z^2)}
\end{aligned}$$

Note: Mathematica did the integral! The result for the full loop is 4 x this result.

$$\begin{aligned}
B_z &= 4 \cdot \frac{\mu_0 i l}{8\pi} \cdot \frac{8 l \sqrt{(\frac{l}{2})^2 + z^2}}{(l^2 + 2z^2) \cdot (l^2 + 4z^2)} \\
&= \frac{\mu_0 i l}{2\pi} \cdot \frac{8 l \sqrt{(\frac{l}{2})^2 + z^2}}{(l^2 + 2z^2) \cdot (l^2 + 4z^2)}
\end{aligned}$$

4. A circular loop with radius a is centered at the origin. Find the magnetic field at a point $(0,0,z)$. You may assume a current I in the clockwise direction.



We begin by writing \vec{l} , the position of a little bit of current. This will allow us to find $d\vec{l}$.

$$\vec{l} = x'\hat{i} + y'\hat{j} = a\cos\theta\hat{i} + a\sin\theta\hat{j}$$

$$\frac{d\vec{l}}{d\theta} = -a\sin\theta\hat{i} + a\cos\theta\hat{j}$$

$$d\vec{l} = (-a\sin\theta\hat{i} + a\cos\theta\hat{j})d\theta$$

Here the angle θ is measured from the x axis, as usual. The primed coordinates are the coordinates of the wire. Now find the other pieces to plug into the Biot-Savaart law...

$$\begin{aligned}\vec{r} &= (0 - x')\hat{i} + (0 - y')\hat{j} + (z - 0)\hat{k} \\ &= -a\cos\theta\hat{i} - a\sin\theta\hat{j} + z\hat{k}\end{aligned}$$

$$r = \sqrt{(-a\cos\theta)^2 + (-a\sin\theta)^2 + z^2} = \sqrt{a^2 + z^2}$$

$$\hat{r} = \frac{-a\cos\theta\hat{i} - a\sin\theta\hat{j} + z\hat{k}}{\sqrt{a^2 + z^2}}$$

and substitute

$$\begin{aligned}
\vec{B} &= \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{i d\vec{l} \times \hat{r}}{r^2} = \\
&= \frac{\mu_0 i}{4\pi} \int_0^{2\pi} \frac{(-a \sin\theta \hat{i} + a \cos\theta \hat{j}) \times \frac{-a \cos\theta \hat{i} - a \sin\theta \hat{j} + z \hat{k}}{\sqrt{a^2 + z^2}}}{a^2 + z^2} d\theta \\
&= \frac{\mu_0 i}{4\pi} \int_0^{2\pi} \frac{(-a \sin\theta \hat{i} + a \cos\theta \hat{j}) \times (-a \cos\theta \hat{i} - a \sin\theta \hat{j} + z \hat{k})}{(a^2 + z^2)^{3/2}} d\theta \\
&= \frac{\mu_0 i}{4\pi} \int_0^{2\pi} \frac{a^2 \hat{k} + az \cos\theta \hat{i} + az \sin\theta \hat{j}}{(a^2 + z^2)^{3/2}} d\theta \\
B_x &= \frac{\mu_0 i az}{4\pi(a^2 + z^2)^{3/2}} \int_0^{2\pi} \cos\theta d\theta = 0 \\
B_y &= \frac{\mu_0 i az}{4\pi(a^2 + z^2)^{3/2}} \int_{\pi/2}^{2\pi} \sin\theta d\theta = 0 \\
B_z &= \frac{\mu_0 i a^2}{4\pi(a^2 + z^2)^{3/2}} \int_0^{2\pi} d\theta = \frac{\mu_0 i a^2}{2(a^2 + z^2)^{3/2}}
\end{aligned}$$

Note that the directions match what we would expect from the right hand rule...