

The Statistics of Random Events

Statistical Error in Measurement

Introduction

In the experimental pursuit of a quantitative description of the physical world, Physicists are concerned with the quality of their measurements. This concern focuses on the *accuracy* of measurements and on the *precision* of measurements. Accuracy is a measure of how well a measurement matches a “true” or expected value. When a measurement is accurate, it matches the expected value well. Precision is a measure of how close together repeated measurements “cluster.” When repeated measurements are tightly clustered, that is, when the repeated measurements yield almost the same value over and over, we say that the measurement is precise. An error bar or an “uncertainty” is a quantitative measurement of how precise a measurement is.

In this experiment, we will investigate how the precision is measured and how repeated measurement improves the precision by reducing the statistical error. We will also investigate how uncertainties in directly measured quantities propagate to become uncertainties in quantities that are calculated from those directly measured values.

Uncertainty in Measured Quantities--Why is it Important?

Every measured physical quantity has an uncertainty or error associated with it. This uncertainty is every bit as important to the measurement as is the measured value and the units of the measurement. The uncertainty allows us to interpret the meaning of the measurement. Consider the following example. We do an experiment where we measure the mass of a subatomic particle. Our initial measurement of the mass is:

$$m = 8.8 \times 10^{-31} \pm 0.3 \times 10^{-31} \text{ kg}$$

The plus/minus is the uncertainty in the measurement. Ideally this uncertainty is measured, but in some cases, it may be an estimate made by the experimentalist. We know that the electron mass has been measured to high precision elsewhere and it has the value

$$m = 9.1 \times 10^{-31} \text{ kg}$$

Given the uncertainty in our measurement, we can conclude that the particle that we saw was an electron since the mass we measured was within one error bar of the “given” value. Another way of saying this would be to conclude that our particle is “consistent” with an electron. To really conclude that its an electron, we might want to investigate other properties like electric charge, etc.

Our conclusion would be different if the uncertainty in our measurement were different. If the measured mass were

$$m = 8.8 \times 10^{-31} \pm 0.003 \times 10^{-31} \text{ kg}$$

we would say that that the particle that we observed was “inconsistent” with being an electron. We may have discovered a new particle, or we may have something going on in our experiment that leads us to understate the uncertainty. In either case, something interesting is going on that will lead to further experiments and investigations.

Sources and Types of Uncertainty

Uncertainties can be categorized into three groups.

1. Blunders-- Strictly speaking, these are mistakes, not uncertainties. By blunders, we refer to the misuse of a measuring instrument, a calculation that has been done incorrectly, the use of a wrong unit or constant, etc. These are not “allowed” errors. It is assumed that when an experimentalist reports his/her findings, he/she has corrected all of the blunders.

2. Statistical or Random Error--Statistical errors occur naturally in the process of repeating a measurement. For example, if an experimentalist uses a meter stick to measure a length over and over again, it is not possible for him/her to place the meter stick in absolutely the same place each time. If an experimentalist measures a mass on a triple beam balance, it is not possible to place the counter weight in exactly the same place each time.

Measurements that show statistical errors are just as likely to be “too small” as “too large” when compared to the “true” value. We say that the measurements are randomly distributed about the correct value. Because of this property, we can obtain more and more accurate measurements by repeating the measurement over and over again and averaging the results. The precision of the measurement is improved as since the uncertainty in the average decreases with repeated measurement.

3. Systematic Error--Systematic Errors occur because of some problem in the measurement technique that is unknown to the experimenter. These errors often occur because of an instrumental problem. A miscalibrated meter may read values that are systematically too low. A meter stick may have stretched because of some kind of environmental condition. Measurements with systematic errors are biased toward being too low or too high so the systematic uncertainty cannot be reduced with repeated measurement. Systematic errors are the most difficult to discover, control and reduce.

It is important to notice that there is nothing called “human error.” You should *never* use this term. You should work hard to eliminate all the blunders. Remaining errors should be classified as statistical or systematic.

Determining the Uncertainty

In journal quality experiments, an experimentalists would work hard to measure the uncertainty or error in his/her measurements. To do this, the experimentalists would do many, many repetitions of the same measurement while carefully controlling experimental conditions. The number of repetitions can easily reach into the millions. After taking measurements, the experimentalist will average them and then examine how they are distributed about the average or mean value.

To begin, we will consider making repeated measurements of some variable x . We will label each measurement x_i and we will compute the mean. The **mean** is given by

$$x_{avg} = \frac{1}{N} \sum_{i=1}^N x_i \quad (1)$$

where N is the number of measurements and x_i are the measured values.

We can now compute the variance s_x^2 . This is number measures how “spread out” the repeated measurements are--that is, it measures how close measurements are to the sample mean x_i .

$$s_x^2 = \frac{1}{N} \sum_{i=1}^N (x_i - x_{avg})^2 \quad (2)$$

(Baird, p. 44). No matter how many times we repeat the measurement, the sample variance remains the same (excepting natural fluctuations that we assume to be small when N is large). Note that this calculation is done over the entire population of measurements. The variance and standard deviation have a slightly different definition when the calculation is done over some subset of the total population.

The standard deviation is computed from the sample variance:

$$\sigma_x = s_x = \sqrt{s_x^2} \quad (3)$$

When we refer to the “error” in a measurement, we are referring to the sample standard deviation. It measures how precise a measurement is-- it measures how likely repeated measurements are to be near the mean of the set of measurements.

As the number of measurements goes up, the uncertainty in the mean value goes down. This is why we perform repeated measurements. We can reduce the uncertainty in the mean by doing lots of measurements.

In the limit where there are many many measurements and the measurements are distributed in a normal (Gaussian Bell) curve about the mean, If we repeat our measurements over and over, 68% of the measurements will yield a value that is within 1σ of the mean. Similarly, 98% will fall within 2σ and 99% will fall within 3σ . If an mean value is more than three standard deviations away from the value that we expect to be the “true” result (like our measurement of the the mass of the electron), we say that our measured value is not consistent with our expected value.

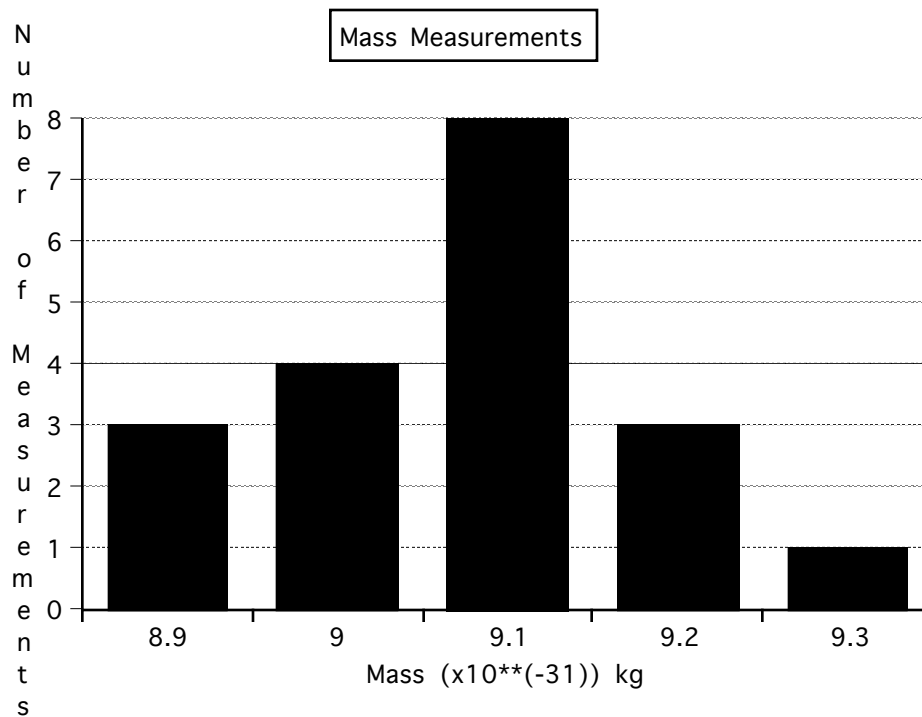
In the experiments that we will do, it will often not be possible to take enough data to meaningfully determine the standard deviation by equating it with the variance. N will often be too small. In light of this difficulty, we will often estimate the size of the uncertainty. The uncertainty in a measurement x will be denoted as Δx . Ideally this number would be the sample standard deviation (which would equal the standard deviation), but for our purpose it will be our best determination of the uncertainty, whether it be calculated or estimated.

An Example.

The table below shows data that we might obtain if we measured the mass of an electron twenty times.

| Trial | Mass ($\times 10^{-31}$ kg) | Trial | Mass ($\times 10^{-31}$ kg) |
|-------|------------------------------|-------|------------------------------|
| 1 | 9.15 | 11 | 9.25 |
| 2 | 9.10 | 12 | 9.10 |
| 3 | 9.05 | 13 | 9.05 |
| 4 | 9 | 14 | 9.15 |
| 5 | 8.90 | 15 | 8.95 |
| 6 | 9.10 | 16 | 9.20 |
| 7 | 9.15 | 17 | 9.10 |
| 8 | 9.20 | 18 | 9.05 |
| 9 | 9.25 | 19 | 8.90 |
| 10 | 9.30 | 20 | 9.15 |

The mean value is 9.11×10^{-31} kg and the uncertainty is $\sigma = 0.11 \times 10^{-31}$ kg . If we “bin” the data, we can begin to see a bell curve forming about the average value.



Notice that a rough “bell” curve is forming in these data.

Propagation of Error

Often the physical quantity that we are interested in is not what we measure directly. We may wish to know the speed, but we measure distance and time and compute the speed. We need to address the question of how the uncertainties in the directly measured quantities (raw measurements) lead to (or propagate) to become uncertainties in the calculated quantity. We will consider that we have measured three quantities and we know the uncertainty, either by measurement or estimate. These

quantities may be also be averages of measurements and the uncertainties are the uncertainties in the average:

$$a \pm \Delta a$$

$$b \pm \Delta b$$

$$c \pm \Delta c$$

Addition/Subtraction-- If we add or subtract quantities with uncertainties, we compute the uncertainty that propagates to the sum or difference

$$s = a + b + c$$

$$\Delta s = \sqrt{(\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2}$$

$$d = a + b - c$$

$$\Delta d = \sqrt{(\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2}$$

Multiplication/Division-- When we multiply or divide using numbers with uncertainties, the fractional error propagates in a way similar to simple addition and subtraction

$$p = a \cdot b \cdot c$$

$$\frac{\Delta p}{p} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$$

$$\Delta p = p \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$$

$$q = \frac{a \cdot b}{c}$$

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$$

$$\Delta q = q \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2}$$

Combinations of operations-- When combinations of operations are used, we calculate errors by applying the rules discuss above systematically to each part of the calculation

$$f = \frac{a}{b} + c$$

$$\Delta\left(\frac{a}{b}\right) = \frac{a}{b} \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}$$

$$\Delta f = \sqrt{\left(\Delta\left(\frac{a}{b}\right)\right)^2 + (\Delta c)^2} = \sqrt{\left(\frac{a}{b}\right)^2 \cdot \left[\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2\right] + (\Delta c)^2}$$

Special Functions--We also need to be able to propagate errors in special functions, like how the uncertainty in an angle propagates through the sine function, or how raising a

measured quantity to a power propagates. For a function $f(a)$ we can calculate the uncertainty in Δf :

$$\Delta f(a) = \left. \frac{\partial f}{\partial a} \right|_a \Delta a$$

More generally, we can write:

$$f = f(a, b, c, \dots)$$

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial a}\right)^2 (\Delta a)^2 + \left(\frac{\partial f}{\partial b}\right)^2 (\Delta b)^2 + \left(\frac{\partial f}{\partial c}\right)^2 (\Delta c)^2 + \dots}$$

There is a second way to find the uncertainty in a computed quantity. Rather than using the formulae above, you may also simply plug in and recalculate the function using the ranges of measured values.

Lets consider an example. Consider the function f with argument $a = 70^\circ$, $\Delta a = 3^\circ$ We proceed by converting to radians first, and then computing.

$$f = \sin(a) = 0.9397$$

$$\Delta f = \cos(a) \cdot \Delta a = 0.342 \cdot 0.05236 = 0.0179$$

$$f = 0.9397 \pm 0.0179$$

Now lets try computing using the range of arguments:

$$f(a) = 0.9379$$

$$f(a + \Delta a) = 0.9563 (\Delta f = +0.0184)$$

$$f(a - \Delta a) = 0.9205 (\Delta f = -0.0174)$$

Notice while the that the second method yields an uncertainty that is roughly the same as the first, the uncertainty that is asymmetric. This is because of the shape of the sine function. This should cause us to be rather careful about interpreting results that are inconsistent at the 3σ level.

Procedure

In this experiment, we will study two systems that have random errors associated with them. In the first case, we will examine the distribution of hits of a ball dropped on a target. In the first case, we will drop a ball on a target. Because of unavoidable problems in releasing the ball, not every drop will land in the center of the target. A distribution will be formed. In the second case, we will examine the decay of a radioactive element. Radioactive decay is a random process and we will observe the distribution function that forms about the average number of decays per second that occurs.

Detailed Procedure

Target--Part A

1. Take a piece of standard white typing paper and draw lines to divide it in half top to bottom and right to left. Place a dot at the center. This dot will be the target.
2. Place the paper on top of a piece of carbon paper so that the carbon side faces up. Place the paper and carbon paper on the floor near a lab bench.
3. Drop a wooden/metal ball onto the paper from approximately one meter (table-top height) and catch the ball before it bounces. You should not spend any time aiming from above. You should reposition your arm on each try--don't simply lock in one places When the ball strikes the paper, it will live an imprint so that its contact point can be determined. You should do 30 hits on a piece of paper and then repeat with a second sheet to get 60 hits total.
4. Measure the distance of each hit from one of the halfway lines and record.
5. Compute the average position from your data using equation (1).
6. Compute the uncertainty in the position using equation (2).
7. Bin your data and construct a histogram. Test to see if 68% fall within one uncertainty.
8. The uncertainty that we are measuring actually reflects the uncertainty in the angle of release as well as the position of your hand. Let us assume that the uncertainty primarily comes from the angle. Using the uncertainty that you have determined experimentally, propagate the error back to find the uncertainty in the angle of release.
- 9.* Check your hand calculation using a computer or calculator program.

Target--Part B

1. Using the same data sheet as in part A, now measure the distance from the central target point to each impact point. Now each measurement will be positive.
2. Compute the average distance from the central point.
3. Bin the data and plot the data. What does the distribution look like? Does the mean that you computed correspond to the peak in your graph?

Nuclear Statistics*

1. While you are taking data in the Target section, your instructor will be recording data from a radioactive sample. He/she will record the number of counts that were measured over a 30 second interval
2. Compute the average number of decays that occur in a 30 second period using equation (1). *Note: you are encouraged to use a computer/calculator to do this.*
3. Compute the uncertainty in the number of decays using equation (2). *Note: you are encouraged to use a computer/calculator to do this.*
4. Bin your data and examine the distribution. Do 68% of the events fall within 1 uncertainty. Do 95% fall within two uncertainties.
5. Compute the average number of counts per second. This is a measure of the "activity" of the source.
6. Compute the uncertainty in the activity that arises from the uncertainty in the number of counts (as you have already determined) and the uncertainty in the measurement in time. You will need to estimate the uncertainty in the time.

**To be done out of lab.*

Questions

1. Are the distributions that you observed “Gaussian” (Normal) distributions? What tells you this? Why would you expect this? If they are not, why not?
2. As you increase the number of observations, what do you expect will happen to the distribution function. Sketch how the distributions will change as more and more observations are added.
3. Are there any systematic errors present in these experiments? How could they be controlled, evaluated, or eliminated.
4. Given the uncertainty in the release angle, at what distance is one sigma if the ball is dropped from 2 m? Do you need a larger piece of paper?

References:

Experimentation: An Introduction to Measurement Theory and Experiment

Design by D. C. Baird

Data Reduction and Error Analysis for the Physical Sciences by Philip R. Bevington

Experiments in Modern Physics by Adrian C. Melissinos