Sample Exam 4

Short Answer

1. A metal bar of length 1 m travels through a perpendicular B field with magnitude 1 T. How fast would the bar need to go to develop an emf of 1 V across its ends.

$$\varphi_B = lBx$$
$$\varepsilon = -\frac{d\varphi_B}{dt} = -lB\frac{dx}{dt}$$
$$\varepsilon = -lBv$$
$$|\,v\,| = \frac{\varepsilon}{lB} = \frac{1V}{1m \cdot 1T} = 1m/s$$

2. A uniform magnetic field passes through a square loop of wire with 5 turns. The loop has side 0.1 m and is oriented at 30 degrees with respect to the field. What is the flux through the wire? If the field drops from 5 T to 4 T in 1 minute, what emf will develop and in what direction will it point.

$$\varphi_{1loop} = BA\cos\theta = 5T \cdot (0.1)^2 \cdot \cos 30 = 0.0433Tm^2$$
$$\varphi = 5 \cdot \varphi_{1loop} = 0.217Tm^2$$
$$|\varepsilon| = \frac{d\varphi}{dt} = 5 \cdot \frac{(5T - 4T)\cdot(0.1)^2 \cdot \cos 30}{60s} = 7.22 \times 10^{-4}V$$

3. A cylindrical solenoid is 0.5 m long and has 50 turns per cm. What field is present if 1 amp flows through the wire? (Hint: Calculate using Ampere's Law). What is the energy density?

$$\oint \vec{B} \cdot d\vec{l} = \mu_0\, i_{enc}$$
$$Bl = \mu_0\, i\, n\, l$$
$$B = \mu_0 in = 4\pi \times 10^{-7} \cdot 1A \cdot \frac{50}{0.01m} = 6.283 \times 10^{-3}T$$
$$u = \frac{1}{2\mu_0}B^2 = \frac{1}{2 \cdot 4\pi \times 10^{-7}} \cdot (6.283 \times 10^{-3}T)^2$$

4. Compute the inductance of the solenoid in SA 3.

$$L = \frac{\varphi}{i} = \frac{NBA}{i} = \frac{nl\mu_0 i\, n\, A}{i} = \mu_0 n^2 l A$$
$$L = 4\pi \times 10^{-7} \cdot (\frac{50}{0.01m})^2 \cdot 0.5m \cdot A$$

5. A 100mH inductor has a current given by $i=2t^2+3t+4$. What voltage develops across the inductor at t=3 sec.

$$V_L = L\frac{di}{dt} = L\cdot(4t+3) = 100\times10^{-3}\cdot(4\cdot3+3) = 1.5V$$

6. Write Maxwell's equations and briefly explain each equation.

$$(a)\quad \oint \vec{E}\cdot d\vec{A} = \frac{q_{encl}}{\varepsilon_0}$$

$$(b)\quad \oint \vec{E}\cdot d\vec{r} = -\frac{d\varphi_B}{dt}$$

$$(c)\quad \oint \vec{B}\cdot d\vec{A} = 0$$

$$(d)\quad \oint \vec{B}\cdot d\vec{r} = \mu_0 i_{encl} + \mu_0\varepsilon_0\frac{d\varphi_E}{dt}$$

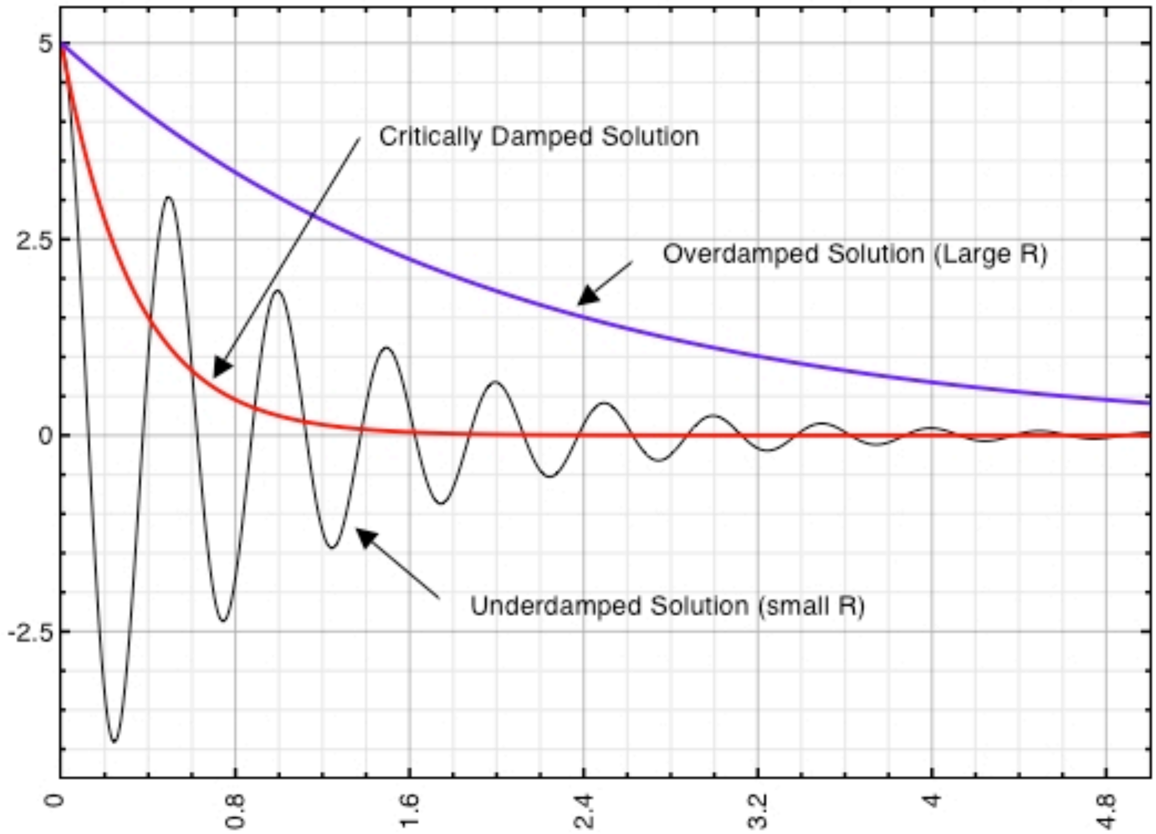
(a) Gauss' Law for E: The total flux of E through an enclosed surface is proportional to the charge enclosed

(b) A changing magnetic flux induces an electric field.

(c) Gauss' Law for B: There is no free magnetic charge.

(d) A magnetic field is produced by either a current or a changing electric flux.

7. Describe the underdamped, critically damped, and overdamped solutions of the RLC circuit.



8. An emf with frequency 60 Hz and $V_0 = 300$ drives a circuit with resistance 100 Ohms and inductance 0.5 H. What are the RMS voltage, the inductive reactance X_L, the impedance Z, and the maximum and RMS current.

$$V_{RMS} = \frac{V_0}{\sqrt{2}} = 212.1V$$
$$X_L = \omega L = 2\pi f L = 2\pi \cdot 60Hz \cdot 0.5H = 60\pi\,\Omega = 188.5\,\Omega$$
$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100\,\Omega)^2 + (188.5\,\Omega - 0)^2} = 213.4\,\Omega$$
$$i_{RMS} = \frac{V_{RMS}}{Z} = \frac{212.1V}{213.4\,\Omega} = 0.994A$$
$$i_0 = \sqrt{2}\, i_{RMS} = 1.41A$$

10. An incoming radio wave has a frequency f of 780 Khz. If the inductor in an LC circuit is 100mH, what should the capacitance be set to to tune this radio to this frequency.

$$f = 780 \times 10^3\, Hz$$

$$2\pi f = \sqrt{\frac{1}{LC}}$$

$$C = \frac{1}{L(4\pi^2 f^2)} = \frac{1}{100 \times 10^{-3}(4\pi^2 \cdot (780 \times 10^3\, Hz)^2)} = 4.16 \times 10^{-13} F$$

10. Derive the expression for the magnitude of the magnetic field at a distance r from a long straight wire.

We use Ampere's Law

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\, i$$

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\, i$$
$$B \cdot (2\pi r) = \mu_0\, i$$
$$B = \frac{\mu_0\, i}{2\pi r}$$

11. A current of 10A is distributed uniformly across a beam with a radius a. Find the magnetic field at a distance r from the center of the beam. Consider both r<a and r>a.
We can now use Ampere's Law (J is the current density: current /area)

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\, i$$

For r<a

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\, i$$
$$B \cdot (2\pi r) = \mu_0\, J\, \pi r^2$$
$$B = \frac{\mu_0\, J\, r}{2}$$

For r>=a

$$\oint \vec{B} \cdot d\vec{r} = \mu_0\, i$$
$$B \cdot (2\pi r) = \mu_0\, J\, \pi a^2$$
$$B = \frac{\mu_0\, J\, a^2}{2r}$$

Problems.

1. A magnetic field is given by

$$B = B_0 \cos\omega t$$

Note that the magnitude of B changes, but not its direction. A circular loop is placed in this field and oriented at an angle θ

a. What is the magnetic flux through the loop?

$$\varphi = BA\cos\theta$$

b. What emf is induced? Draw a picture to indicate the direction of the induced current

$$\begin{aligned}
\varepsilon &= -\frac{d\varphi_B}{dt} \\
&= -\pi r^2 \frac{dB}{dt}\cos\theta \\
&= -\pi r^2(-\omega B_0 \sin\omega t)\cos\theta \\
&= \pi r^2 \omega B_0 \sin\omega t \cos\theta
\end{aligned}$$

The emf oscillates back and forth! The direction changes back and forth, always flowing to produce a B to oppose the change.

c. What angle leads to the maximum induced emf?

$\theta = 0$ leads to maximum change in flux

d. Assume that θ is now 0. What electric field is present at the radius of the wire?

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{l} &= -\frac{d\varphi_B}{dt} \\
E(2\pi r) &= -\pi r^2 \frac{dB}{dt} \\
E &= -\frac{r}{2}\cdot(-\omega B_0 \sin\omega t) \\
E &= \frac{r}{2}\cdot(\omega B_0 \sin\omega t)
\end{aligned}$$

2. A circular parallel plate capacitor with radius R and separation d in an RC circuit discharges via the equation

$$q = q_0 e^{-t/RC}$$

a. What is the electric field as a function of time?

$$V = \frac{q}{C} = \frac{q_0 e^{-t/RC}}{C} \qquad C = \frac{\varepsilon_0 A}{d}$$

$$E = \frac{V}{d} = \frac{q_0 e^{-t/RC}}{Cd}$$

b. What is the displacement current at a/2 and a?

$$\begin{aligned} i_d &= \varepsilon_0 \frac{d\varphi_E}{dt} \\ &= \varepsilon_0 (\pi a^2) \frac{dE}{dt} \\ &= \varepsilon_0 (\pi a^2) \cdot \frac{q_0 e^{-t/RC}}{Cd} \cdot \frac{-1}{RC} \quad at \;\; radius \;\; a \\ &= \varepsilon_0 (\pi \frac{a^2}{4}) \cdot \frac{q_0 e^{-t/RC}}{Cd} \cdot \frac{-1}{RC} \quad at \;\; radius \;\; a/2 \end{aligned}$$

c. What is the induced B at at a/2 and a?

$$\begin{aligned} &\oint \vec{B} \cdot d\vec{l} = \mu_0 i_d \\ &B(2\pi a) = \mu_0 i_d \\ &B = \frac{\mu_0 i_d}{(2\pi a)} \; at \; radius \; a \\ &B = \frac{\mu_0 i_d}{(2\pi \frac{a}{2})} \; at \; radius \; r/2 \end{aligned}$$

Note: You need to plug in the correct displacement current from part b.

3. Consider an RLC circuit with R=10 Ohms, L = 500 mH and C=1microF.

a. Write the differential equation that describes this circuit by using Kirchoff's voltage loop rule. What is the natural osc. frequency?

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{500\times 10^{-3}\,H\cdot 1\times 10^{-6}\,F}} = 1414.2$$

b. Now consider driving this circuit. What are X_L,X_C, and Z for this circuit? Sketch the amplitude of the voltage across the capacitor as a function of the frequency.

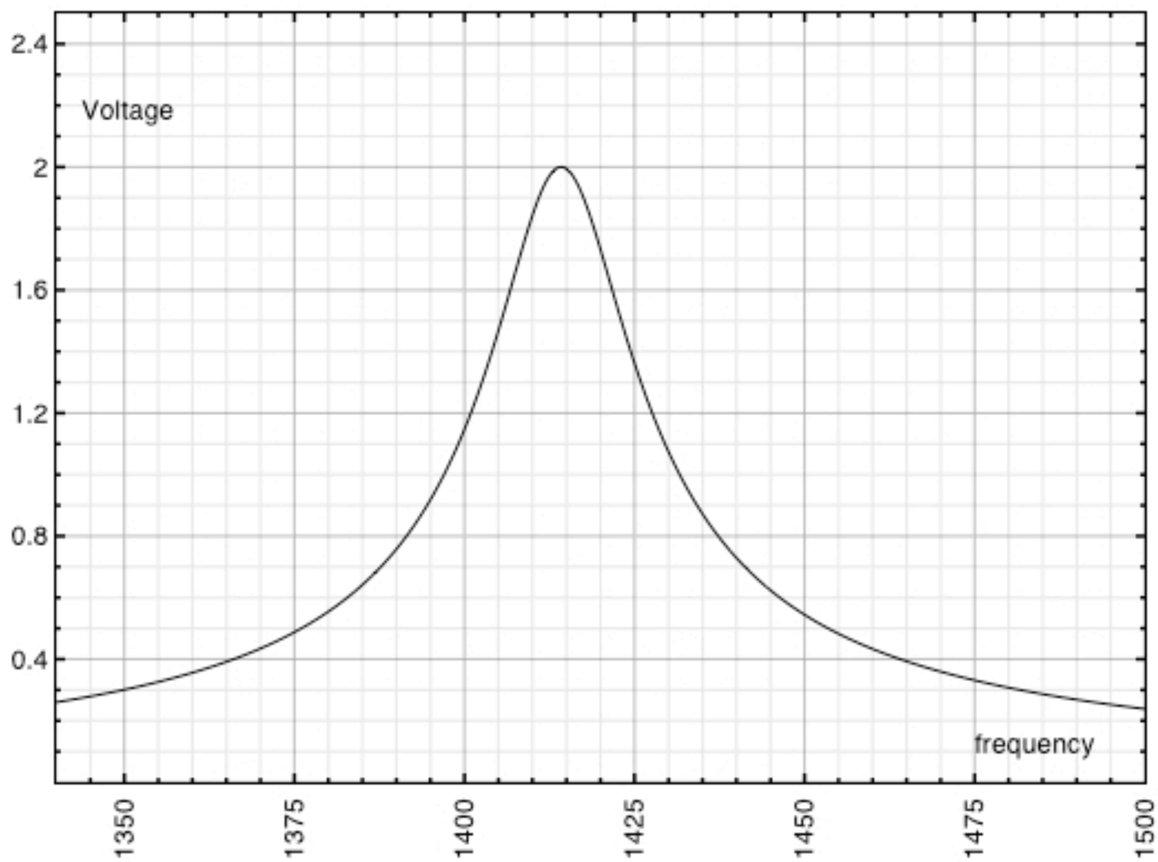
$$X_L = \omega L = 2\pi f L$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

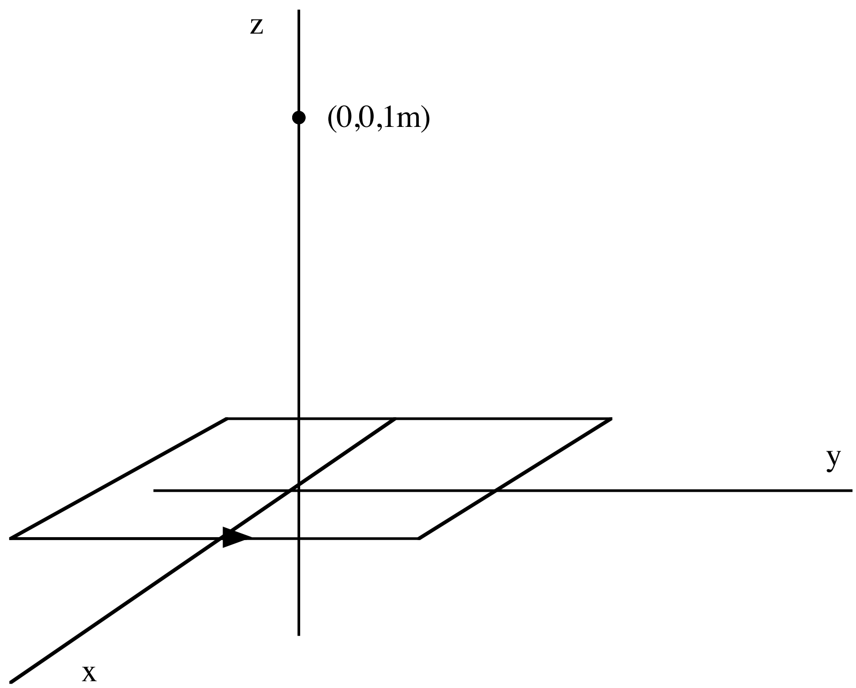
$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

Take the emf to be 20 for the purpose of plotting a nice curve. Here y is the voltage and x is the frequency.

$$y = \frac{20.0}{\sqrt{\left(10^2 + \left(x\cdot 0.5 - \frac{1}{x\cdot 1\text{E-}6}\right)^2\right)}}$$



3. A Square loop with side 1 m contains a single turn of wire. It is placed so that the center of the square is at the origin. Compute the magnetic field at a point (0,0,1m). Take the current to be 5 Amps and flowing in the counterclockwise direction as viewed from above.



The field can be computed by finding the z component due to just one of the four segments and then multiplying by 4. By symmetry, only the z component survives. We begin by computing the field using Biot-Savart for the ***right*** wire shown above. First, write out all the pieces we need.

$$d\vec{l} = dx\,\hat{i}$$

$$\vec{r} = (0-x)\,\hat{i} + (0-\frac{l}{2})\,\hat{j} + (z-0)\,\hat{k}$$

$$r = \sqrt{(0-x)^2 + (0-\frac{l}{2})^2 + (z-0)^2}$$

$$\hat{r} = \frac{-x\,\hat{i} - \frac{l}{2}\,\hat{j} + z\,\hat{k}}{\sqrt{x^2 + (\frac{l}{2})^2 + z^2}}$$

Note that the right wire is at y=l/2. The direction of the current will be given by the order of integration (from x=+l/2 to x=-l/2).

Now substitute in

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{l/2}^{-l/2} \frac{i\,d\vec{l}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\int_{l/2}^{-l/2} \frac{i(dx\hat{i})\times\left(\dfrac{-x\,\hat{i} - \frac{l}{2}\hat{j} + z\,\hat{k}}{\sqrt{x^2+(\frac{l}{2})^2+z^2}}\right)}{x^2+(\frac{l}{2})^2+z^2}$$

$$= \frac{\mu_0}{4\pi}\int_{l/2}^{-l/2} \frac{i(dx\,\hat{i})\times(-x\,\hat{i} - \frac{l}{2}\hat{j} + z\,\hat{k})}{(x^2+(\frac{l}{2})^2+z^2)^{3/2}}$$

We now pick out just the z component (from i x j)

$$B = \frac{\mu_0}{4\pi}\int_{l/2}^{-l/2} \frac{i(dx\,\hat{i})\times(-x\,\hat{i} - \frac{l}{2}\hat{j} + z\,\hat{k})}{(x^2+(\frac{l}{2})^2+z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 i}{4\pi}\int_{l/2}^{-l/2} \frac{(-\frac{l}{2}dx\ \hat{k})}{(x^2+(\frac{l}{2})^2+z^2)^{3/2}}$$

$$= -\frac{\mu_0 i\,l}{8\pi}\int_{l/2}^{-l/2} \frac{dx}{(x^2+(\frac{l}{2})^2+z^2)^{3/2}}$$

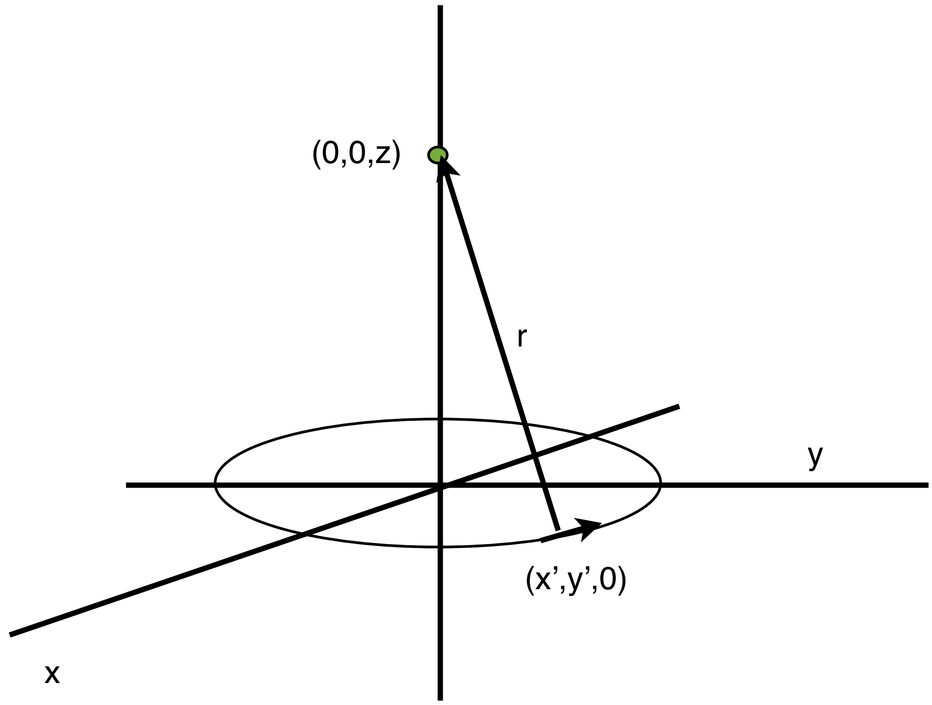
$$= \frac{\mu_0 i\,l}{8\pi}\cdot\frac{8\,l\sqrt{(\frac{l}{2})^2+z^2}}{(l^2+2z^2)\cdot(l^2+4z^2)}$$

Note: Mathematica did the integral! The result for the full loop is 4 x this result.

$$B_z = 4\cdot\frac{\mu_0 i\,l}{8\pi}\cdot\frac{8\,l\sqrt{(\frac{l}{2})^2+z^2}}{(l^2+2z^2)\cdot(l^2+4z^2)}$$

$$= \frac{\mu_0 i\,l}{2\pi}\cdot\frac{8\,l\sqrt{(\frac{l}{2})^2+z^2}}{(l^2+2z^2)\cdot(l^2+4z^2)}$$

4. A circular loop with radius a is centered at the origin. Find the magnetic field at a point (0,0,z). You may assume a current I in the clockwise direction.



We begin by writing l, the position of a little bit of current. This will allow us to find dl.

$$\vec{l} = x'\hat{i} + y'\hat{j} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$$
$$\frac{d\vec{l}}{d\theta} = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j}$$
$$d\vec{l} = (-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j})\,d\theta$$

Here the angle theta is measured from the x axis, as usual. The primed coordinates are the coordinates of the wire. Now find the other pieces to plug into the Biot-Savaart law...

$$\vec{r} = (0 - x')\hat{i} + (0 - y')\hat{j} + (z - 0)\hat{k}$$
$$= -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\,\hat{k}$$
$$r = \sqrt{(-a\cos\theta)^2 + (a\sin\theta)^2 + z^2} = \sqrt{a^2 + z^2}$$
$$\hat{r} = \frac{-a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\,\hat{k}}{\sqrt{a^2 + z^2}}$$

and substitute

$$\vec{B} = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{i\,d\vec{l}\times\hat{r}}{r^2} =$$

$$= \frac{\mu_0 i}{4\pi}\int_0^{2\pi}\frac{(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j})\times\dfrac{-a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\,\hat{k}}{\sqrt{a^2+z^2}}}{a^2+z^2}d\theta$$

$$= \frac{\mu_0 i}{4\pi}\int_0^{2\pi}\frac{(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j})\times(-a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\,\hat{k})}{(a^2+z^2)^{3/2}}d\theta$$

$$= \frac{\mu_0 i}{4\pi}\int_0^{2\pi}\frac{a^2\hat{k} + az\cos\theta\,\hat{i} + az\sin\theta\,\hat{j}}{(a^2+z^2)^{3/2}}d\theta$$

$$B_x = \frac{\mu_0 iaz}{4\pi(a^2+z^2)^{3/2}}\int_0^{2\pi}\cos\theta\,d\theta = 0$$

$$B_y = \frac{\mu_0 iaz}{4\pi(a^2+z^2)^{3/2}}\int_{\pi/2}^{2\pi}\sin\theta\,d\theta = 0$$

$$B_z = \frac{\mu_0 ia^2}{4\pi(a^2+z^2)^{3/2}}\int_0^{0}d\theta = \frac{\mu_0 ia^2}{2(a^2+z^2)^{3/2}}$$

Note that the directions match what we would expect from the right hand rule...